

Geometric approach to nonvariational singular elliptic equations

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Abstract

In this work we develop a systematic geometric approach to study fully nonlinear elliptic equations with singular absorption terms as well as their related free boundary problems. The magnitude of the singularity is measured by a negative parameter $(\gamma - 1)$, for $0 < \gamma < 1$, which reflects on lack of smoothness for an existing solution along the singular interface between its positive and zero phases. We establish existence as well sharp regularity properties of solutions. We further prove that minimal solutions are non-degenerate and obtain fine geometric-measure properties of the free boundary $\mathfrak{F} = \partial\{u > 0\}$. In particular we show sharp Hausdorff estimates which imply local finiteness of the perimeter of the region $\{u > 0\}$ and \mathcal{H}^{n-1} a.e. weak differentiability property of \mathfrak{F} .

1 Introduction

The aim of this present work is to study fine qualitative properties of nonvariational singular elliptic equations of the form

$$(1.1) \quad \begin{cases} F(D^2u) \sim u^{-\theta} \cdot \chi_{\{u>0\}} & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $\theta = 1 - \gamma$, for $0 < \gamma < 1$, f is a continuous, non-negative boundary datum and the governing operator F is assumed to be uniform elliptic, i.e., $(D_{(i,j)}F)_{1 \leq i,j \leq N}$ is a positive definite matrix. The study of singular equations as (1.1) is motivated by applications in a number of problems in engineering sciences. In fact the free boundary problem

$$(1.2) \quad \begin{cases} F(D^2u) = \gamma u^{\gamma-1} & \text{in } \{u > 0\} \\ u = |\nabla u| = 0 & \text{on } \partial\{u > 0\} \end{cases}$$

is used, for example, to model fluids passing through a porous body Ω . For instance, u could represent the density of a gas, or else the density of certain chemical specie, in reaction with a porous catalyst pellet, Ω .

The variational theory, $F(M) = \text{Trace}(M)$, for the free boundary problem (1.2) is fairly well understood, nowadays. It appears as the Euler-Lagrange equation in the minimization of non-differentiable functionals:

$$(1.3) \quad \int \frac{1}{2} |\nabla u(X)|^2 + u(X)^\gamma dX \longrightarrow \min.$$

See, for instance [15, 16, 2, 21]. Notice that such a problem is quite different from the one treated in the classical paper [7]. The latter has been recently studied in the fully nonlinear setting in [9].

The case $\gamma = 1$ in (1.3) represents the obstacle problem, [4]; the case $\gamma = 0$ relates to the cavitation problem, [1]. Fully nonlinear version of the obstacle problem has been considered in [13]. Nonvariational cavitation problem has been recently studied in [17]. The delicate intermediary case, $0 < \gamma < 1$, addressed in this present work brings major novelty adversities as the equation satisfied within the positive set $\{u > 0\}$ is nonhomogeneous and blows-up along the *a priori* unknown quenching interface $\mathfrak{F} = \partial\{u > 0\} \cap \Omega$ - the so called free boundary of the problem. The lack of variational or energy approaches too implies significant difficulties in the problem and new, nonvariational solutions have to be established. In fact, since the free boundary problem considered in this paper has nonvariational character, one cannot use the powerful measure-distributional language to setup weak version of the problem. Instead we shall employ a perturbation scheme and will obtain uniform estimates with respect to the approximating parameter ε . A solution to the fully nonlinear free boundary problem (1.2) will therefore be obtained as the limit of appropriate approximating configurations.

The first main problem to be addressed concerns the optimal regularity for solutions to Equation (1.1). Optimal estimates for heterogeneous equations, $Lu = f(X, u)$ is in general a quite delicate issue. For the singular setting studied in this present work, optimal estimates are even more involved as they can be understood as invariant (tangential) equations for their own scaling. We show in Section 4 of the present work that solutions are locally of class $C^{1, \frac{\gamma}{2-\gamma}}$. This result was only known in the variational setting, for minimizers of Euler-Lagrange functional, see [15, 16, 2] and [10, 11].

The second principal result devilered in this article states that minimal solutions, i.e., solutions obtained from Perron's type method do grow precisely as $\text{dist}(X, \mathfrak{F})^{1+\frac{\gamma}{2-\gamma}}$, which corresponds to the maximum growth rate allowed. Such a result implies a quite restrictive geometry for the free quenching interface \mathfrak{F} . As consequence of our sharp gradient estimate, Theorem 4.1 and optimal growth rate, Theorem 5.4, a minimal solution is trapped between the graph of two multiples of $\text{dist}(X, \mathfrak{F})^{1+\frac{\gamma}{2-\gamma}}$, i.e.,

$$\underline{c} \cdot \text{dist}(X, \mathfrak{F})^{1+\frac{\gamma}{2-\gamma}} \leq u(X) \leq \overline{C} \cdot \text{dist}(X, \mathfrak{F})^{1+\frac{\gamma}{2-\gamma}}, \quad X \in \{u > 0\}.$$

By means of geometric considerations, in Section 6 we establish a *clean* Harnack inequality for solutions to (1.1) within free boundary tangential balls, $B \subset \{u > 0\}$, B tangent to \mathfrak{F} . In Section 7, under an extra asymptotic structural assumption on the governing operator F , we establish Hausdorff estimates of the free boundary. In particular we show $\chi_{\{u > 0\} \cap \Omega'} \in \text{BV}(\Omega)$, that is, $\{u > 0\}$ is locally a set of finite perimeter. We further show that the reduced free boundary has \mathcal{H}^{n-1} total measure. The last two Sections close up the project by obtaining a solution to the fully nonlinear free boundary problem (1.2) with the desired analytic and geometric properties.

2 Mathematical set-up

Throughout this paper Ω will be a fixed Lipschitz bounded domain in \mathbb{R}^N , $f: \partial\Omega \rightarrow \mathbb{R}_+$ is a continuous boundary datum and $0 < \gamma < 1$ is a fixed real number. We shall denote by $\text{Sym}(N)$ the space of all real $N \times N$ symmetric matrices and $F \in C^1(\text{Sym}(N) \setminus \{0\})$ will be a uniformly elliptic fully nonlinear operator; that is, we shall assume that there exist two constants $0 < \lambda \leq \Lambda$ such that

$$(2.1) \quad F(\mathcal{M} + \mathcal{N}) \leq F(\mathcal{M}) + \Lambda \|\mathcal{N}^+\| - \lambda \|\mathcal{N}^-\|, \quad \forall \mathcal{M}, \mathcal{N} \in \text{Sym}(N).$$

The ultimate goal of this paper is to study existence and fine qualitative properties of solutions to the singular equation

$$(2.2) \quad F(D^2u) = \gamma u^{\gamma-1} \cdot \chi_{\{u>0\}}.$$

From the equation itself, one notices that the Hessian of an existing solution blows-up along the free boundary $\mathfrak{F} = \partial\{u > 0\} \cap \Omega$; therefore, solutions cannot be of class C^2 . In the fully nonlinear setting, the problem of optimal regularity for solutions to Equation (2.2) is a rather delicate issue and it will be addressed in Section 4. Part of the subtleness of this problem comes from the intrinsic complexity of the regularity theory for viscosity solutions to uniform elliptic equations. We recall that it is well known that solutions to homogeneous equation

$$(2.3) \quad F(D^2u) = 0,$$

has a priori $C^{1,\mu}$ bounds for some $\mu > 0$ that depends only on N, λ and Λ . Under concavity or convexity assumption on F , a Theorem due to Evans and Krylov, states that solutions are $C^{2,\alpha}$. Nevertheless, Nadirashvili and Vladut have recently shown that given any $0 < \eta < 1$ it is possible to build up a uniformly elliptic operator F , whose solutions to the homogeneous equation (2.3) are not $C^{1,\eta}$, see [14], Theorem 1.1.

Therefore, in order to access the optimal regularity estimate available for the free boundary problem (2.2), it is natural to assume that F has a priori $C^{2,\tau}$ estimates for some small $0 < \tau < 1$. Such a hypothesis will be enforced hereafter in the paper, though all but Theorem 4.1 do not depend on such condition.

Let us turn our attention to the singularly perturbed approach we shall use in order to grapple with the lack of variational approaches available. In this paper we suggest the following singular perturbation scheme to appropriately approach the free boundary problem (1.2):

$$(E_\varepsilon) \quad \begin{cases} F(D^2u) &= \beta_\varepsilon(u), & \text{in } \Omega \\ u &= f & \text{on } \partial\Omega. \end{cases}$$

The singular perturbation term β_ε is build up as follows: initially select your favorite function $\rho \in C_0^\infty[0, 1]$ and set

$$(2.4) \quad \alpha := 1 + \frac{\gamma}{2 - \gamma}.$$

Throughout the whole paper, α will always be the fixed value stated in (2.4). In the sequel, define

$$(2.5) \quad B_\varepsilon(t) = \int_0^{\frac{t-\varepsilon^\alpha \sigma_0}{\varepsilon^\alpha}} \rho(s) ds,$$

where $0 < \sigma_0 < \frac{1}{2}$ is an arbitrary technical choice. Notice that B_ε is a smooth approximation of $\chi_{(0,\infty)}$. Finally, we set

$$(2.6) \quad \beta_\varepsilon(t) = \gamma t^{\gamma-1} B_\varepsilon(t).$$

Such a construction is carefully carried out as to preserve the natural scaling of the desired equation (2.2).

We finish this Section by listing the main notations adopted throughout the article:

- The dimension of the Euclidean space the problem is modeled in will be denoted by $N \geq 2$. Ω will be a fixed bounded domain in \mathbb{R}^N . For a domain $\mathcal{O} \subset \mathbb{R}^N$, $\partial \mathcal{O}$ will represent the boundary of the domain \mathcal{O} . χ_S will stand for the characteristic function of the set S .
- The N -dimensional Lebesgue measure of a set $A \subset \mathbb{R}^N$ will be denoted by $\mathcal{L}^N(A)$. \mathcal{H}^{n-1} will stand for the $(n-1)$ -Hausdorff measure.
- $\langle \cdot, \cdot \rangle$ will be the standard scalar product in \mathbb{R}^N . For a vector $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, its Euclidean norm will be denoted by $|\xi| := \sqrt{\langle \xi, \xi \rangle}$. The tensor product $\xi \otimes \psi$ denotes the matrix whose entries are given by $\xi_i \psi_j$ for $1 \leq i, j \leq N$.
- $B_r(p)$ will be the open ball centered at p with radius r . Furthermore, we shall denote $kB = kB_r(p) := B_{kr}(p)$, for any $k > 0$.
- Constants $C, C_1, C_2, \dots > 0$ and $c, c_0, c_1, c_2, \dots > 0$ that depend only on dimension, γ and ellipticity constants λ, Λ will be call universal. Any additional dependence will be emphasized.

3 Existence of minimal solutions

In this section we comment on the existence of a viscosity solution to equation (E_ε) . More importantly, we shall establish herein a stable process to select special solutions to (E_ε) . As we will show in Section 5, the family of minimal solutions turns out to satisfy the desired appropriate geometric features. Such properties will allow us to establish Hausdorff estimates of the free boundary in Section 7.

Notice that because of the lack of monotonicity of equation (E_ε) with respect to the variable u , classical Perron's method cannot be directly employed. The next theorem proved in [17], is an adaptation of Perron's method, which is by now fairly well understood.

Theorem 3.1. *Let g be a bounded, Lipschitz function defined in the real line \mathbb{R} . Suppose F uniformly elliptic and that the equation $F(D^2u) = g(u)$ admits a Lipschitz viscosity subsolution u_\star and a Lipschitz viscosity supersolution u^\star such that $u_\star = u^\star = f \in C(\partial\Omega)$. Define the set of functions,*

$$S := \{w \in C(\overline{\Omega}); \quad u_\star \leq w \leq u^\star \quad \text{and } w \text{ supersolution of } F(D^2u) = g(u)\}.$$

Then,

$$v(x) := \inf_{w \in S} w(x)$$

is a continuous viscosity solution of $F(D^2u) = g(u)$ and $v = f$ continuously on $\partial\Omega$.

Existence of minimal solution to Equation (E_ε) follows by choosing $u_\star = u_\star(\varepsilon)$ and $u^\star = u^\star(\varepsilon)$ solutions to the following boundary value problems

$$\begin{cases} F(D^2u_\star) = \zeta, & \text{in } \Omega \\ u_\star = f & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} F(D^2u^\star) = 0, & \text{in } \Omega \\ u^\star = f & \text{on } \partial\Omega, \end{cases}$$

where

$$\zeta := \sup \beta_\varepsilon \sim \varepsilon^{\gamma-1}.$$

The existence the functions u_\star and u^\star is consequence of standard Perron's method. By construction u_\star is viscosity subsolution of (E_ε) and u^\star is a viscosity supersolution oh (E_ε) . Note that $u^\star, u_\star \in C^{0,1}(\Omega) \cap C(\overline{\Omega})$. Thus a direct application of Theorem 3.1 yields the following existence result:

Theorem 3.2 (Existence of minimal solutions). *Let $\Omega \in \mathbb{R}^n$ be a Lipschitz domain and $f \in C(\partial\Omega)$ be a nonnegative boundary datum. Then, for each $\varepsilon > 0$ fixed, equation (E_ε) has a nonnegative minimal viscosity solution $u_\varepsilon \in C(\overline{\Omega})$.*

As previously mentioned, more importantly than assuring existence of a viscosity solution to (E_ε) , Theorem 3.2 provides a particular choice of solutions to such an equation. In comparison with the variational theory, this choice is a replacement for the selection of minimizers of the Euler-Lagrange functional (see for instance [19] for further details). Therefore, unless otherwise stated, whenever we mention viscosity solution to (E_ε) , we mean the minimal solution provided by Theorem 3.2.

4 Sharp regularity estimates

The first main result we prove in this paper is the optimal regularity estimate, uniform in ε , available for solutions to (E_ε) . We will show that u_ε is locally a $C^{1,\beta}$ function and we shall further determine the optimal $\beta > 0$ in terms of the degree of singularity γ . This key information has only been known for variational solutions, [15, 10, 11] and the proofs make decisive use of energy considerations. In principle it is not even clear that one should expect the same regularity theory for nonvariational problems.

Thus, we start off this Section by rather informal, heuristic considerations as to guide us through the genuine results to be established later on. Let us analyze the limiting free boundary problem (1.2). Suppose 0 is a free boundary point and, say, $-e_n$ is the unit outward normal

pointing towards the quenching phase $\{u = 0\}$. If u is $C^{1,\beta}$ at 0, then, in a small neighborhood, say, $B_\rho \cap \{u > 0\}$, $\rho \ll 1$, u behaves like $\sim X_n^{1+\beta}$. Therefore, the singular potential of the equation in (1.2) is like $\sim X_n^{(1+\beta) \cdot (1-\gamma)}$. In view of the regularity theory for heterogeneous fully nonlinear equations $F(D^2u) = f(X)$, established in [5] and [20], we obtain the following implication

$$X_n^{(1+\beta) \cdot (1-\gamma)} \in L_{\text{weak}}^\theta \quad \text{implies} \quad u \in C^{1, 1 - \frac{1}{\theta}}.$$

The reasoning above gives the following system of algebraic equations

$$\begin{cases} \theta(1+\beta)(\gamma-1) &= -1 \\ \beta &= 1 - \frac{1}{\theta}. \end{cases}$$

Solving for β , reveals, $\beta = \frac{\gamma}{2-\gamma}$, which agrees with the optimal regularity estimate established for the variational theory.

This Section is devoted to establish local $C^{1, \frac{\gamma}{2-\gamma}}$ regularity estimates for solutions u_ε to Equation (E_ε) , uniform in ε . Recall that we are working under the natural assumption that F has *a priori* $C^{2,\tau}$ estimates. In fact we shall obtain a universal control on the gradient of u_ε near the free boundary in terms of the value of u_ε . Since $u_\varepsilon = 0$ along the free boundary, our estimate gives the desired regularity through the interface $\mathfrak{F} = \partial\{u > 0\} \cap \Omega$. Actually the gradient estimates we obtain are even stronger than the aimed $C^{1, \frac{\gamma}{2-\gamma}}$ regularity. Here is its precise statement:

Theorem 4.1 (Uniform optimal regularity). *Given a subset $\Omega' \Subset \Omega$, there exists a constant C depending on, $\|f\|_\infty$, γ , Ω' , dimension, ellipticity, but independent of ε , such that, any family of viscosity solutions $\{u_\varepsilon\}$ of equation (E_ε) satisfies,*

$$|\nabla u_\varepsilon(X)|^2 \leq C u_\varepsilon(X)^\gamma, \quad \forall X \in \Omega'.$$

In particular, $u_\varepsilon \in C_{\text{loc}}^{1, \frac{\gamma}{2-\gamma}}$, uniformly in ε .

Proof. For simplicity, we shall drop the subscript ε in u_ε , writing simply u . We will analyze the following auxiliary function

$$v := \psi(u_\varepsilon) |\nabla u_\varepsilon|^2, \quad \text{for } \psi(t) = t^{-\gamma}.$$

Our ultimate goal is to show that v is locally bounded in Ω , for bounds that do not depend on ε . Hereafter in the proof we select a positive function $\phi \in C^2(\Omega)$ that vanishes on $\partial\Omega$ and satisfies $|\nabla \phi|^2 = O(\phi)$. The purpose of such a function is merely to localize our analysis. Define, in the sequel,

$$\omega := \phi \cdot v \quad \text{in } \Omega,$$

and let $X_0 \in \Omega$ be a maximum point of ω in $\bar{\Omega}$, that is

$$\phi(X_0) \cdot v(X_0) = \max_{\bar{\Omega}} \omega.$$

Differentiating ω and Equation (E_ε) , we obtain

$$(4.1) \quad D_i \omega = \phi_i v + \phi v_i, \quad D_{ij} \omega = \phi_{ij} v + \phi_i v_j + \phi_j v_i + \phi v_{ij}$$

and

$$(4.2) \quad \sum_{i,j} F_{ij}(D^2 u) D_{ij} u_k = \beta'_\varepsilon(u) u_k.$$

Let $A_{ij} := F_{ij}(D^2 u(X_0))$. By uniform ellipticity of the operator F , the matrix (A_{ij}) is strictly positive. Also, since X_0 is a maximum point, $D^2 \omega(X_0)$ is non-positive. Therefore

$$(4.3) \quad \begin{aligned} 0 &\geq \sum_{i,j} A_{ij} D_{ij} \omega(X_0) \\ &= \left[v \sum_{i,j} A_{ij} D_{ij} \phi + 2 \operatorname{Tr}((A_{ij}) \nabla \phi \otimes \nabla v) + \phi \sum_{i,j} A_{ij} D_{ij} v \right] (X_0). \end{aligned}$$

It follows from (4.1) and from the fact that X_0 is a critical point of ω that

$$(4.4) \quad \nabla v(X_0) = -v(X_0) \frac{\nabla \phi(X_0)}{\phi(X_0)}.$$

Combining (4.4), ellipticity of (A_{ij}) and analytic properties of ϕ , we reach

$$(4.5) \quad \begin{aligned} v \sum_{i,j} A_{ij} D_{ij} \phi + 2 \operatorname{Tr}((A_{ij}) \nabla \phi \otimes \nabla v) &\geq - \left| v \sum_{i,j} A_{ij} D_{ij} \phi + 2 \operatorname{Tr}((A_{ij}) \nabla \phi \otimes \nabla v) \right| \\ &= -v \left| \sum_{i,j} A_{ij} D_{ij} \phi + \frac{2}{\phi} \operatorname{Tr}((A_{ij}) \nabla \phi \otimes \nabla \phi) \right| \\ &\geq -C(\Lambda) \max_{\bar{\Omega}} \left\{ |D^2 \phi|, \frac{|\nabla \phi|^2}{\phi} \right\} v \\ &=: -C_0 v. \end{aligned}$$

Let us turn our attention on the term $\sum_{i,j} A_{ij} D_{ij} v(X_0)$. Differentiating v , we obtain

$$(4.6) \quad D_i v = \psi'(u) u_i |\nabla u|^2 + 2\psi(u) \sum_k u_k u_{ki}.$$

Differentiating above expression, one reaches

$$\begin{aligned} D_{ij} v &= (\psi'' u_i u_j + \psi'(u) u_{ij}) |\nabla u|^2 + 2\psi'(u) u_i \sum_k u_k u_{kj} \\ &\quad + 2\psi'(u) u_j \sum_k u_k u_{ki} + 2\psi(u) \sum_k (u_{kj} u_{ki} + u_k u_{kij}). \end{aligned}$$

It follows from (4.4) and (4.6) that, at X_0 , for each i ,

$$\sum_k u_k u_{ki} = -\frac{1}{2\psi(u)} \left\{ \psi'(u) u_i |\nabla u|^2 + v \frac{\phi_i}{\phi} \right\}.$$

Thus, combining these with (4.1), we find

$$\begin{aligned}
 A_{ij}D_{ij}v &= \left[\psi''(u) - 2\frac{(\psi'(u))^2}{\psi(u)} \right] A_{ij}\nabla u \otimes \nabla u |\nabla u|^2 - 2\frac{v}{\phi} \frac{\psi'(u)}{\psi(u)} A_{ij}\nabla u \otimes \nabla \phi \\
 (4.7) \quad &+ \psi'(u) A_{ij}u_{ij} |\nabla u|^2 + 2\psi(u) \left(\text{Tr}(D^2u(A_{ij})D^2u) + \beta'_\varepsilon(u) |\nabla u|^2 \right).
 \end{aligned}$$

By ellipticity and definition of the ϕ , follows the estimates

$$\begin{aligned}
 A_{ij}\nabla u \otimes \nabla u &\geq \lambda |\nabla u|^2, \\
 |A_{ij}\nabla u \otimes \nabla \phi| &\leq \Lambda |\nabla u| \cdot |\nabla \phi|, \\
 |A_{ij}u_{ij}| &\leq \Lambda F(D^2u) = \Lambda \beta_\varepsilon(u), \\
 \text{Tr}(D^2u(A_{ij})D^2u) &\geq 0, \\
 \left[\psi''(u) - 2\frac{(\psi'(u))^2}{\psi(u)} \right] &= (-\gamma^2 + \gamma)u^{-\gamma-2}.
 \end{aligned}$$

Here it is important to notice that $-\gamma^2 + \gamma > 0$. Using all these above estimates in (4.7) we obtain

$$\begin{aligned}
 A_{ij}D_{ij}v &\geq (-\gamma^2 + \gamma)\lambda v u^{-2} |\nabla u|^2 - 2\gamma \frac{\Lambda v}{u \phi} |\nabla u| \cdot |\nabla \phi| \\
 &\quad - \gamma \Lambda u^{-\gamma-1} \beta_\varepsilon(u) |\nabla u|^2 + 2u^{-\gamma} \beta'_\varepsilon(u) |\nabla u|^2.
 \end{aligned}$$

On the other hand, for $t > 0$

$$\beta_\varepsilon(t) = \gamma t^{\gamma-1} B_\varepsilon(t) \leq \gamma t^{\gamma-1}$$

and

$$\begin{aligned}
 \beta'_\varepsilon(t) &= \gamma(\gamma-1)t^{\gamma-2} B_\varepsilon(t) + \gamma t^{\gamma-1} B'_\varepsilon(t) \\
 &\geq \gamma(\gamma-1)t^{\gamma-2} B_\varepsilon(t).
 \end{aligned}$$

Here we have used

$$B'_\varepsilon(t) = \varepsilon^{-\beta} \rho \left(\frac{t - \sigma_0 \varepsilon^\beta}{\varepsilon^\beta} \right) > 0.$$

These together give us,

$$\begin{aligned}
 A_{ij}D_{ij}v &\geq (-\gamma^2 + \gamma)\lambda v \cdot u^{-2} |\nabla u|^2 - 2\gamma \frac{\Lambda v}{u \phi} |\nabla u| \cdot |\nabla \phi| \\
 (4.8) \quad &- \Lambda \gamma^2 u^{-\gamma-1} \cdot u^{\gamma-1} |\nabla u|^2 - 2\gamma(1-\gamma)u^{\gamma-2} \cdot u^{-\gamma} |\nabla u|^2 \\
 &= (-\gamma^2 + \gamma)\lambda v \cdot u^{-2} |\nabla u|^2 - 2\gamma \frac{\Lambda v}{u \phi} |\nabla u| \cdot |\nabla \phi| \\
 &- \Lambda \gamma^2 u^{-2} |\nabla u|^2 - 2\gamma(1-\gamma)u^{-2} |\nabla u|^2.
 \end{aligned}$$

Combining (4.3), (4.5) and (4.8), taking into account that $|\nabla\phi| = O(\phi)$, we reach

$$\begin{aligned} C_0 v &\geq \phi(\lambda(-\gamma^2 + \gamma)v - C_1)u^{-2}|\nabla u|^2 - 2\gamma\Lambda u^{-1}v|\nabla u| \cdot |\nabla\phi| \\ &= \phi(\lambda(-\gamma^2 + \gamma)v - C_1)u^{-2}|\nabla u|^2 - 2\gamma\Lambda u^{-1}u^{-\gamma}u^{\gamma/2}\sqrt{v}|\nabla\phi| \cdot |\nabla u|^2 \\ &\geq \phi(\lambda(-\gamma^2 + \gamma)v - C_1)u^{-2}|\nabla u|^2 - 2C(\phi)\gamma\Lambda u^{-\frac{\gamma}{2}-1}\sqrt{v\phi}|\nabla u|^2, \end{aligned}$$

where $C_1 := (\gamma^2\Lambda + 2\gamma(1 - \gamma)) > 0$. Clearly we can assume

$$|\nabla u(X_0)|u(X_0) \neq 0.$$

Hence, as $|\nabla\phi|^2 = O(\phi)$, we derive

$$(4.9) \quad C_0 u^{-\gamma+2} \geq \phi(\lambda(-\gamma^2 + \gamma)v - C_1) - 2C(\phi)\gamma\Lambda u^{-\frac{\gamma}{2}+1}\sqrt{v\phi}.$$

In the region $|u_\varepsilon| \geq 1$, $F(D^2u_\varepsilon)$ is uniformly bounded, independently of ε . Thus, by Alexandroff-Bakeman-Pucci maximum principle,

$$|u_\varepsilon| \leq C_2,$$

for a constant C_2 that does not depend upon ε . By such considerations and (4.9) follows

$$C_3 \geq \lambda(-\gamma^2 + \gamma)\phi(X_0)v(X_0) - C_4\sqrt{v(X_0)\phi(X_0)}.$$

for universal constants C_3, C_4 that does not depend upon ε . Clearly the above estimative implies that

$$v(X)\phi(X) \leq v(X_0)\phi(X_0) \leq C,$$

i.e.,

$$\phi u^{-\gamma}|\nabla u|^2 \leq C.$$

for a constant C that depends only on dimension, ellipticity, γ , $\|f\|_\infty$ and ϕ , but is independent of ε .

It is now classical to obtain $\|u_\varepsilon\|_{C^{1, \frac{\gamma}{2-\gamma}}}$ is locally bounded, uniformly in ε . The proof of Theorem 4.1 is concluded. \square

The uniform optimal regularity established in Theorem 4.1 gives, in particular, compactness of the family of solutions to Equation (E_ε) . It will also be important to our analysis the following consequence of Theorem 4.1:

Corollary 4.2. *Given a subdomain $\Omega' \Subset \Omega$, there exist constants C and $r_0 > 0$ depending on γ , Ω' and universal parameters such that for $X_0 \in \Omega'$ and $r \leq r_0$, there holds*

$$\sup_{B_r(X_0)} u_\varepsilon \leq u_\varepsilon(X_0) + Cu_\varepsilon(X_0)^{\gamma/2}r + Cr^\alpha$$

where, $\alpha = 1 + \frac{\gamma}{2-\gamma}$.

Proof. Define the auxiliary function

$$f(Y) := u_\varepsilon(Y) - u_\varepsilon(X_0) - \nabla u_\varepsilon(X_0) \cdot (Y - X_0).$$

where $Y \in B_r(X_0)$. Clearly

$$f(X_0) = |\nabla f(X_0)| = 0$$

and therefore, from Theorem 4.1 we obtain

$$|f(Y) - f(X_0)| \leq C \cdot |Y - X_0|^\alpha,$$

which immediately gives, by triangular inequality,

$$u_\varepsilon(Y) \leq u_\varepsilon(X_0) + |\nabla u_\varepsilon(X_0)| \cdot |Y - X_0| + C|Y - X_0|^\alpha.$$

However, applying once more Theorem 4.1, we reach

$$u_\varepsilon(Y) \leq u_\varepsilon(X_0) + C u_\varepsilon(X_0)^{\gamma/2} \cdot |Y - X_0| + C|Y - X_0|^\alpha,$$

and the proof of Corollary 4.2 is concluded. \square

5 Nondegeneracy of minimal solutions

In the previous Section we have shown that solutions to Equation (E_ε) are locally of class $C^{1, \frac{\gamma}{2-\gamma}}$. In particular such an estimate provides an upper bound on how fast u_ε grows away from, say, the level surface $\{u_\varepsilon \sim \varepsilon^\alpha\}$, for α as in (2.4). That is,

$$u_\varepsilon(Z) \lesssim [\text{dist}(Z, \{u_\varepsilon \sim \varepsilon^\alpha\})]^\alpha.$$

The main result we shall prove in this Section states that minimal solutions do growth precisely as $\text{dist}(X_0, \{u_\varepsilon \sim \varepsilon^\alpha\})^\alpha$, see Corollary 5.5 for the precise statement. In fact we shall establish a stronger nondegeneracy property of minimal solutions, which also has fundamental importance in our blow-up analysis.

To simplify the statement of the results, we introduce some definitions and notations. Hereafter we shall use systematically the following notations:

$$\begin{aligned} \{u_\varepsilon > \kappa\} &:= \{x \in \Omega \mid u_\varepsilon(x) > \kappa\}, \\ \{\tau > u_\varepsilon > \lambda\} &:= \{x \in \Omega \mid \tau > u_\varepsilon(x) > \lambda\}, \\ d_\varepsilon(X) &:= \text{dist}(X, \partial\{u_\varepsilon > \varepsilon^\alpha\}), \end{aligned}$$

The nondegeneracy feature of minimal solutions are based on the construction of appropriate viscosity supersolution whose value within an inner disk is much smaller than its value on the boundary of an outer disk.

Proposition 5.1. *Assume, with no loss of generality that $0 \in \Omega$. Given $0 < \eta$, there exists a radially symmetric function $\theta \in C^{1,1}(\Omega)$ and universal small constants $0 < c_2 < 1$ and $0 < c_1 < 1$ such that*

1. $\theta \equiv 2\sigma_0$ in $B_{c_1\eta}$
2. $\theta \geq c_2\eta^{1+\frac{\gamma}{2-\gamma}}$ in $\Omega \setminus B_\eta$
3. θ satisfies $F(D^2\theta(X)) \leq \beta(\theta(X))$, pointwise in Ω , where $\beta = \beta_1$, as in (2.6).

Proof. Initially define

$$\theta(X) = \begin{cases} 2\sigma_0 & \text{for } 0 \leq |X| \leq c_1\eta; \\ a_0(|X| - c_1\eta)^2 + 2\sigma_0 & \text{for } c_1\eta \leq |X| \leq \eta; \\ A|X|^\alpha + B & \text{for } |X| \geq \eta. \end{cases}$$

where the constants a_0, A, B e c_1 will be chosen later. Our first goal is to enforce that such a function is indeed $C^{1,1}$. For this, we have to set along $|X| = \eta$,

$$(5.1) \quad a_0(1 - c_1)^2\eta^2 + 2\sigma_0 = \theta(X) = A\eta^\alpha + B$$

thus, easily we obtain

$$a_0 = \frac{1}{(1 - c_1)^2} [A\eta^{\alpha-2} + \eta^{-2}(B - 2\sigma_0)].$$

Moreover, differentiating θ and matching its gradient along $|X| = \eta$, we obtain

$$(5.2) \quad 2a_0(1 - c_1)X_i = A\alpha\eta^{\alpha-2}X_i.$$

Combining (5.1) and (5.2) we find

$$(5.3) \quad \frac{A\alpha\eta^{\alpha-2}}{2(1 - c_1)} = \frac{1}{(1 - c_1)^2} [A\eta^{\alpha-2} + \eta^{-2}(B - 2\sigma_0)].$$

In the sequel, take

$$c_1 := \frac{\gamma}{2} \in (0, 1),$$

which implies the relation $\alpha = \frac{1}{1-c_1}$, where, as always, α is the value set in (2.4). Finally we set

$$B = 2\sigma_0 - \frac{A}{2}\eta^\alpha,$$

as to (5.3) to be satisfied. Summing up the construction so far, we have built up

$$\theta(X) = \begin{cases} 2\sigma_0 & \text{for } 0 \leq |X| \leq c_1\eta; \\ \frac{A\alpha^2}{2}\eta^{\alpha-2}(|X| - c_1\eta)^2 + 2\sigma_0 & \text{for } c_1\eta \leq |X| \leq \eta; \\ A|X|^\alpha + \left(2\sigma_0 - \frac{A}{2}\eta^\alpha\right) & \text{for } |X| \geq \eta. \end{cases}$$

which is of $C^{1,1}$, by the construction itself. We still have the parameter A to be adjusted later. Our next step is to show that θ is an appropriate supersolution, that is, we want to establish

$$(5.4) \quad F(D^2\theta) \leq \beta(\theta)$$

pointwise. To this end, we first analyze the equation in the region $c_1\eta \leq |X| \leq \eta$. Direct computations yield

$$\theta_{ij} = A\alpha^2\eta^{\alpha-2} \left[\frac{X_i X_j}{|X|^2} + \left(1 - \frac{c_1\eta}{|X|}\right) \left(\delta_{ij} - \frac{X_i X_j}{|X|^2} \right) \right],$$

within $c_1\eta \leq |X| \leq \eta$. At a point of the form $\bar{X} = (|X|, 0, \dots, 0)$, we find out

$$\begin{aligned} \theta_{11} &= A\alpha^2\eta^{\alpha-2} \\ \theta_{ii} &= A\alpha^2\eta^{\alpha-2} \left(1 - \frac{c_1\eta}{|X|}\right) \quad \text{if } i > 1 \\ \theta_{ij} &= 0 \quad \text{if } i \neq j. \end{aligned}$$

By symmetric invariance of θ and ellipticity of F , we obtain

$$(5.5) \quad F(D^2\theta(X)) \leq \Lambda \left[A\alpha^2\eta^{\alpha-2} + (N-1)A\alpha^2\eta^{\alpha-2} \left(1 - \frac{c_1\eta}{|X|}\right) \right] \leq \Lambda N A\alpha^2\eta^{\alpha-2}.$$

Recall N is the dimension of the space. However, within the region $c_1\eta \leq |X| \leq \eta$, we have

$$2\sigma_0 \leq \theta(X) \leq \frac{A}{2}\eta^\alpha + 2\sigma_0.$$

Taking into account that the function $B = B_1$ set in (2.5) is non-decreasing, we readily obtain

$$\begin{aligned} \beta(\theta(X)) &\geq \gamma\theta(X)^{\gamma-1}B(2\sigma_0) \\ &\geq \gamma\theta(\eta)^{\gamma-1}B(2\sigma_0) \\ &\geq \gamma\left(\frac{A}{2}\eta^\alpha + 2\sigma_0\right)^{\gamma-1}B(2\sigma_0). \end{aligned}$$

Therefore, taking $0 < A \ll 1$ small enough,

$$\gamma\left(\frac{A}{2}\eta^\alpha + 2\sigma_0\right)^{\gamma-1}B(2\sigma_0) > \frac{1}{2}\gamma(2\sigma_0)^{\gamma-1}B(2\sigma_0) > \Lambda N A\alpha^2\eta^{\alpha-2}.$$

and we indeed obtain the desired pointwise inequality

$$F(D^2\theta) \leq \beta(\theta(X)),$$

in the region $c_1\eta \leq |X| \leq \eta$. Let us turn our attention to the region $\eta \leq |X|$. Readily we have

$$\theta_{ij} = A\alpha \left[(\alpha-2)|X|^{\alpha-4}X_i X_j + \delta_{ij}|X|^{\alpha-2} \right].$$

Thus, at a point of the form $(|X|, 0, \dots, 0)$, we obtain

$$\begin{aligned} \theta_{11} &= A\alpha(\alpha-1)|X|^{\alpha-2} \\ \theta_{ii} &= A\alpha|X|^{\alpha-2} \quad \text{if } i > 1 \\ \theta_{ij} &= 0 \quad \text{if } i \neq j. \end{aligned}$$

Therefore, again by symmetric invariance of θ and ellipticity of F , we can write

$$(5.6) \quad F(D^2\theta(X)) \leq \Lambda[A\alpha(\alpha-1) + (N-1)A\alpha]|X|^{\alpha-2} \leq \Lambda N A \alpha \eta^{\alpha-2}.$$

On the other hand, in the region $\eta \leq |X|$, we have for $M \geq \sup_{X \in \Omega} |X|$, that

$$M^\alpha \geq |X|^\alpha - \frac{\eta^\alpha}{2} > 0,$$

and so,

$$\beta(\theta(X)) \geq \gamma \left(A \left(|X|^\alpha - \frac{1}{2} \eta^\alpha \right) + 2\sigma_0 \right)^{\gamma-1} B(\theta(\eta)) > \gamma (AM^\alpha + 2\sigma_0)^{\gamma-1} B(\theta(\eta)),$$

Thus, adjusting $A > 0$ even smaller, if necessary, we can assure

$$AM^\alpha + 2\sigma_0 < 4\sigma_0,$$

and therefore,

$$\beta(\theta(X)) > \gamma(4\sigma_0)^{\gamma-1} B(\theta(\eta)).$$

Finally by 5.6 and the inequality above, as well as diminishing the value of $A > 0$ even further, if necessary, we reach

$$F(D^2\theta(X)) \leq \Lambda N A \alpha |X|^{\alpha-2} < \gamma(4\sigma_0)^{\gamma-1} B(\theta(\eta)) \leq \beta(\theta(X)).$$

So its follow (3). By construction (2) is valid, and the proof of Proposition 5.1 follows. \square

Proposition 5.1 provides the existence of the appropriate barrier in the unit scale $\varepsilon = 1$. To furnish the desired supersolution for any $\varepsilon > 0$ small we argue as follows. Fixed $\varepsilon > 0$, we consider the fully nonlinear elliptic operator

$$F_\varepsilon(\mathcal{M}) := \varepsilon^{2-\alpha} F(\varepsilon^{\alpha-2} \mathcal{M}).$$

It is standard to verify that F_ε is uniform elliptic with the same ellipticity constants as F . Proposition 5.1 applied to F_ε provides a $C^{1,1}$ function $\theta = \theta(\varepsilon)$ that satisfies the differential inequality

$$F_\varepsilon(D^2\theta(X)) = \varepsilon^{2-\alpha} F(\varepsilon^{\alpha-2} D^2\theta(X)) \leq \beta_1(\theta(X)).$$

Finally, we define

$$(5.7) \quad \theta_\varepsilon(X) := \varepsilon^\alpha \theta(\varepsilon^{-1} X),$$

where once more, α is the value set in (2.4). We verify readily that θ_ε defined above satisfies

$$\checkmark \quad \theta_\varepsilon = 2\sigma_0 \varepsilon^\alpha \text{ in } B_{c_1 \varepsilon \eta};$$

$$\checkmark \quad \theta_\varepsilon \geq c_2 \eta^\alpha \text{ in } \Omega \setminus B_{\varepsilon \eta};$$

✓ $\theta_\varepsilon \in C^{1,1}(\Omega)$ and it is a supersolution to (E_ε) .

We are ready to establish strong nondegeneracy of minimal solutions to the singularly perturbed problem (E_ε) .

Theorem 5.2 (Strong Nondegeneracy). *Let $X_0 \in \{u_\varepsilon > \varepsilon^\alpha\}$. There exist two universal positive constants $c_0 > 0$ and $r_0 > 0$ such that if $r < r_0$, there holds*

$$\sup_{B_r(X_0)} u_\varepsilon \geq c_0 r^\alpha,$$

for α as in (2.4).

Proof. Given $r < r_0$, we construct θ_ε for $\eta = r/\varepsilon$. By minimality of u_ε ,

$$u_\varepsilon(Z) > \theta_\varepsilon(Z),$$

for some point $Z \in \partial B_r(X_0)$. Indeed, suppose for the sake of contradiction that $u_\varepsilon \leq \theta_\varepsilon$ along ∂B_r . Define

$$w_\varepsilon = \begin{cases} \min\{\theta_\varepsilon, u_\varepsilon\} & \text{in } \overline{B_r}; \\ u_\varepsilon & \text{in } \Omega \setminus \overline{B_r}. \end{cases}$$

Thus, w_ε is supersolution to (E_ε) ; however in $B_{c_1 r}$, we have,

$$u_\varepsilon > \varepsilon^\alpha > 2\sigma_0 \varepsilon^\alpha \equiv \theta_\varepsilon = w_\varepsilon,$$

which contradicts the minimality of u_ε . In conclusion,

$$c_2 r^\alpha \leq \theta_\varepsilon(Z) < u_\varepsilon(Z) \leq \sup_{B_r} u_\varepsilon,$$

and the Theorem is proven. \square

An immediate Corollary of Theorem 5.2 combined with Corollary 4.2 is the upper and lower control of u_ε by r^α in $B_r \subset \{u_\varepsilon > \varepsilon^\alpha\}$.

Corollary 5.3. *Given a subdomain $\Omega' \Subset \Omega$, there exists a universal constant $C = C(\Omega')$ such that for $X_0 \in \Omega' \cap \{u_\varepsilon > \varepsilon^\alpha\}$ and $r \leq r_0$,*

$$C^{-1} r^\alpha \leq \sup_{B_r(X_0)} u_\varepsilon \leq u_\varepsilon(X_0) + C u_\varepsilon(X_0)^{\gamma/2} r + C r^\alpha$$

Recall we have set the following notation: $d_\varepsilon(X) = \text{dist}(X, \partial\{u_\varepsilon > \varepsilon^\alpha\})$. Our next step is to show that in fact u_ε does growth at the sharp rate away from the free boundary, that is $\sim d_\varepsilon^\alpha$.

Theorem 5.4 (Sharp Growth). *Let $X_0 \in \{u_\varepsilon > \varepsilon^\alpha\}$. Then there exists $c_0 > 0$ universal such that*

$$u_\varepsilon(X_0) \geq c_0 d_\varepsilon(X_0)^\alpha.$$

Proof. Let us suppose for sake of contradiction that no such a constant exists. If so, there would exist a sequence of points $X_n \in \{u_\varepsilon > \varepsilon^\alpha\}$, with $d_n := d_\varepsilon(X_n) \rightarrow 0$ and

$$u_\varepsilon(X_n) \leq \frac{1}{n} d_n^\alpha.$$

Let us define

$$v_n(Y) := \frac{1}{d_n^\alpha} u_\varepsilon(X_n + d_n Y).$$

The function $v_n \geq 0$ in B_1 , and by $C^{1,\alpha-1}$ regularity of u_ε , Theorem 4.1, v_n is bounded in $C_{\text{loc}}^{1,\alpha-1}(B_1)$. Easily we verify that v_n is a minimal solution to

$$(5.8) \quad F_n(D^2 v_n) = \gamma v_n^{1-\gamma} B_{\frac{\varepsilon}{d_n}}(v_n) \quad \text{in } B_1,$$

where $F_n(\mathcal{M}) := d_n^{2-\alpha} F(d_n^{\alpha-2} \mathcal{M})$ for all $\mathcal{M} \in \text{Sym}(N)$ and $B_{\frac{\varepsilon}{d_n}}$ is the smooth approximation of t^+ set up in (2.5). From its very definition, we check that

$$(5.9) \quad B_{\frac{\varepsilon}{d_n}}(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \sigma_0 \left(\frac{\varepsilon}{d_n}\right)^\alpha, \\ 1 & \text{for } t \leq (1 - \sigma_0) \cdot \left(\frac{\varepsilon}{d_n}\right)^\alpha. \end{cases}$$

Since F_n is uniformly elliptic with same ellipticity constants as F , we can apply Theorem 5.2 to v_n as to obtain

$$(5.10) \quad \sup_{B_\kappa} v_n \geq c_0 \kappa^\alpha,$$

for a universal constant c_0 and for any $\kappa > 0$. However, from $C^{1,\alpha-1}$ universal regularity for v_n , see the proof of Corollary (4.2), there holds

$$v_n(X) \leq v_n(0) + C v_n(0)^{\frac{\gamma}{2}} |X| + C |X|^\alpha,$$

for a universal constant $C > 0$. In particular, for $\kappa_0 \ll 1$,

$$v_n(X) \leq v_n(0) + v_n(0)^{\frac{\gamma}{2}} \left(\frac{\sigma_0}{10}\right)^{\frac{1}{\alpha}} \varepsilon + \frac{\sigma_0}{10} \varepsilon^\alpha, \quad \text{in } B_{\kappa_0}.$$

If we take $n \gg 1$, $v_n(0) \leq \frac{9}{10} \sigma_0 \varepsilon^\alpha$ and then

$$v_n(X) \leq \sigma_0 \varepsilon^\alpha \quad \text{in } B_{\kappa_0}.$$

In view of Equation 5.8 and (5.9), we see

$$F_n(D^2 v_n) = 0 \quad \text{in } B_{\kappa_0},$$

for $n \gg 1$. But then, by classical homogeneous Harnack inequality, see [5], and strong nondegeneracy stated in (5.10)

$$c_0 \left(\frac{\kappa_0}{2}\right)^\alpha \leq \sup_{B_{\frac{\kappa_0}{2}}} v_n \leq C v_n(0) = o(1),$$

which finally gives us a contradiction. \square

An important consequence of Theorem 5.4 is the complete control of $u_\varepsilon(X)$ in terms of the $d_\varepsilon(X)^\alpha$.

Corollary 5.5. *Given a subdomain $\Omega' \Subset \Omega$, there exists a universal constant $C = C(\Omega')$ such that for $X \in \Omega' \cap \{u_\varepsilon > \varepsilon^\alpha\}$ and $\varepsilon \leq d_\varepsilon(X)$,*

$$Cd_\varepsilon(X)^\alpha \geq u_\varepsilon(X) \geq C^{-1}d_\varepsilon(X)^\alpha.$$

Proof. The inequality by below is precisely the statement of Theorem 5.4. Now for $Z \in \partial\{u_\varepsilon > \varepsilon^\alpha\}$, it follows from Corollary 4.2

$$u_\varepsilon(X) \leq \varepsilon^\alpha + C\varepsilon^{\alpha\frac{\gamma}{2}}d_\varepsilon(X) + Cd_\varepsilon(X)^\alpha \leq Cd_\varepsilon(X)^\alpha,$$

and the Corollary is proven. \square

As usual a fine geometric control as the one stated in Corollary 5.5 implies uniform positive density of the approximating region $\{u_\varepsilon > \varepsilon^\alpha\}$.

Corollary 5.6. *Given a subdomain $\Omega' \Subset \Omega$, there exists constant $0 < c \leq 1$, depending only on Ω' and universal parameters, such that for any $X \in \Omega' \cap \{u_\varepsilon > \varepsilon^\alpha\}$ and $\varepsilon \ll \delta$, we have*

$$\frac{\mathcal{L}^N(B_\delta(X) \cap \{u_\varepsilon > \varepsilon^\alpha\})}{\mathcal{L}^N(B_\delta)} \geq c.$$

Proof. By strong non-degeneracy there exists $Y_0 \in \overline{B_\delta(X)} \cap \{u_\varepsilon > \varepsilon^\alpha\}$ such that

$$u_\varepsilon(Y_0) \geq c_0\delta^\alpha.$$

By optimal regularity and Corollary 5.3, we obtain, for $Y \in B_{\tau\delta}(Y_0) \cap B_\delta(X)$

$$\begin{aligned} u(Y) &\geq u(Y_0) - |\nabla u(Y_0)| \cdot |Y - Y_0| - C_1|Y - Y_0|^\alpha \\ &\geq c_0\rho^\alpha - C_1\rho^{\alpha\frac{\gamma}{2}}|Y - Y_0| - C_1|Y - Y_0|^\alpha \\ &\geq (c_0 - C_1(\tau + \tau^\alpha))\delta^\alpha \\ &> \varepsilon^\alpha, \end{aligned}$$

provided $0 < \tau \ll 1$ is chosen universally small. In conclusion,

$$\mathcal{L}^N(B_\delta(X) \cap \{u_\varepsilon > \varepsilon^\alpha\}) \geq \mathcal{L}^N(B_\delta(X) \cap B_{\tau\delta}(Y_0)) \geq c\delta^N.$$

for a universal constant $c > 0$. \square

6 Harnack type inequalities

It is well established that Harnack type inequalities are among the central properties of solutions to second order elliptic equations. For non-negative viscosity solutions to fully nonlinear equations with non-homogeneous right hand side,

$$F(D^2v) = f(X), \quad Q_1$$

Krylov-Safonov [12] and Caffarelli [5] (see also [6], Chapter 4) proved the following sharp Harnack inequality:

$$(6.1) \quad \sup_{Q_{1/2}} v \leq C(n, \lambda, \Lambda) \left(\inf_{Q_{1/2}} v + \|f\|_{L^n(Q_1)} \right).$$

As mentioned in previous Sections, one of the major mathematical difficulties in dealing with singular equations as in (1.1) is the fact that right hand side blows-up near the quenching region. In particular, if one tries to interpret the singular term $\gamma u^{\gamma-1}$ as a right hand side $f(X)$ for the equation, classical Harnack inequality (6.1) gives no information near the free boundary.

The key objective of this Section is to establish, uniform-in- ε *clean* geometric Harnack type inequalities for solutions to equation (E_ε) .

Theorem 6.1 (L^1 -Harnack inequality). *Given $\Omega' \Subset \Omega$, $X_0 \in \{u_\varepsilon > \varepsilon^\alpha\} \cap \Omega'$. Then*

$$\int_{B_\rho(X_0)} u_\varepsilon dx \geq c\rho^\alpha,$$

for a universal constant $c > 0$, independent of ε .

Proof. From Lemma 5.2, there is a $Z \in \overline{B_\rho(X_0)} \cap \{u_\varepsilon > \varepsilon^\alpha\}$ and a $c_0 > 0$ universal, such that

$$u_\varepsilon(Y_0) \geq c_0\rho^\alpha.$$

As in the proof of the Corollary 5.6, for $\theta \ll 1$ but universal, we obtain

$$u_\varepsilon(Y) \geq C\rho^\alpha \quad \text{in } B_{\theta\rho}(Y_0).$$

Finally,

$$\int_{B_\rho(X_0)} u_\varepsilon dx \geq C_N \int_{B_\rho(X_0) \cap B_{\theta\rho}(Z)} u_\varepsilon dx \geq C\rho^\alpha,$$

for $C > 0$ a universal constant. Thus, the proof is concluded. \square

Our next Theorem is a *clean* Harnack inequality for ball touching the approximating free boundary $\partial\{u_\varepsilon > \varepsilon^\alpha\}$.

Theorem 6.2 (Harnack Inequality for tangential balls). *Let $X_0 \in \{u_\varepsilon > \varepsilon^\alpha\}$ and $\varepsilon \leq d := d_\varepsilon(X_0)$. Then, there exist a universal constant $C > 0$ such that*

$$\sup_{B_{\frac{d}{2}}(X_0)} u_\varepsilon \leq C \inf_{B_{\frac{d}{2}}(X_0)} u_\varepsilon.$$

Proof. Let $\xi_0, \xi_1 \in \overline{B_{\frac{d}{2}}(X_0)}$, such that

$$\inf_{B_{\frac{d}{2}}(X_0)} u_\varepsilon = u(\xi_0) \quad \text{and} \quad \sup_{B_{\frac{d}{2}}(X_0)} u_\varepsilon = u(\xi_1).$$

As $d_\varepsilon(\xi_0) \geq \frac{d}{2}$, by nondegeneracy

$$(6.2) \quad u_\varepsilon(\xi_0) \geq C_1 d^\alpha.$$

By other hand, using the corollary 5.3, we get

$$u_\varepsilon(\xi_1) \leq u_\varepsilon(X_0) + C_2 u_\varepsilon(X_0)^{\frac{\gamma}{2}} d + C_2 d^\alpha.$$

As in the proof of Corollary 4.2, to $Y \in \partial \{u_\varepsilon > \varepsilon^\alpha\}$, we have that

$$u_\varepsilon(X_0) \leq u_\varepsilon(Y) + C_2 u_\varepsilon(Y)^{\frac{\gamma}{2}} d + C_2 d^\alpha \leq C_3 d^\alpha.$$

So, by the three last inequalities, we obtain

$$\sup_{B_{\frac{d}{2}}(X_0)} u_\varepsilon \leq C \inf_{B_{\frac{d}{2}}(X_0)} u_\varepsilon.$$

for a constant $C > 0$ that does not depend of u_ε and ε . □

7 Hausdorff estimates of the free boundary

In this section, we turn our attention to uniform geometric-measure properties of $\sim \varepsilon^\alpha$ -level surfaces of u_ε . These surfaces approximate the limiting free boundary $\mathfrak{F} := \partial \{u > 0\} \cap \Omega$, where u is the desired limiting function. Through this section we shall work under the following extra structural condition on the operator F :

Definition 7.1. We say a uniformly elliptic operator $F : \text{Sym}(N) \rightarrow \mathbb{R}$ is asymptotically concave if there exists a positive definite matrix $\mathcal{F} = (f_{ij})_{ij}$ and a nonnegative constant $C_F \geq 0$ such that

$$(AC) \quad f_{ij} \mathcal{M}_{ij} - F(\mathcal{M}) \geq -C_F,$$

for all matrix $\mathcal{M} \in \text{Sym}(N)$,

Initially, let us point out that indeed hypothesis (AC) is an asymptotic condition as $\|\mathcal{M}\| \gg 1$, as it suffices to hold in the limit for $\|\mathcal{M}\| \rightarrow +\infty$. It represents a sort of concavity condition at infinity of F . For concave operators, $C_F = 0$. The structural condition (AC) arises from recent considerations on the *recession* operator

$$F^*(\mathcal{M}) := \lim_{\mu \rightarrow 0} \mu F(\mu^{-1} \mathcal{M}).$$

The limiting operator F^* should be interpreted as the tangential equation for the natural elliptic scaling on F . For example, for a number of elliptic operators, it is possible to verify the existence of the limit

$$b_{ij} := \lim_{\|\mathcal{M}\| \rightarrow \infty} F_{ij}(\mathcal{M}).$$

In this case, $F^*(\mathcal{M}) = \text{Tr}(b_{ij}\mathcal{M})$ and (AC) is automatically satisfied. A particularly interesting example is the class of Hessian operators of the form

$$F_{\mathfrak{t}}(M) = f_{\mathfrak{t}}(\lambda_1, \lambda_2, \dots, \lambda_N) := \sum_{j=1}^N (1 + \lambda_j^{\mathfrak{t}})^{1/\mathfrak{t}},$$

where \mathfrak{t} is an odd natural number. For this family of operators, we have $F_{\mathfrak{t}}^* = \Delta$ and condition (AC) is satisfied.

In [17] it is proven that the recession operator F^* rules the free boundary condition for fully nonlinear cavitation problems. In [18], it is established further regularity estimates of solutions to $F(X, D^2u) = f(X)$ via properties of the recession function.

Before continuing, let us make few remarks as to organize some systematic arguments that will appear within the next proofs.

Remark 7.2. Given $X_0 \in \{u_{\varepsilon} > \varepsilon^{\alpha}\}$, where $u_{\varepsilon}(X_0) = C_1 \varepsilon^{\alpha}$ for $C_1 > 1$, $\varepsilon \ll \rho$ and ρ universally small, we have from optimal regularity that in $B_{\rho}(X_0)$, for $\rho \ll 1$ to be adjusted soon, there holds

$$u_{\varepsilon}^{\gamma-1} \geq (C_1 \varepsilon^{\alpha} + C_2 (C_1 \varepsilon^{\alpha})^{\frac{\gamma}{2}} \rho + C_2 \rho^{\alpha})^{\gamma-1}.$$

Therefore, if ε is small as to

$$\varepsilon^{\alpha} < \frac{\rho^{\alpha}}{C_1}$$

and the radius ρ is also selected universally small as to

$$(1 + 2C_2)\rho^{\alpha} \leq \sqrt[\gamma-1]{\frac{2}{\gamma} C_F}$$

we readily obtain

$$\gamma u_{\varepsilon}^{\gamma-1} \geq 2C_F \quad \text{in } B_{\rho}(X_0),$$

for $C_F > 0$ as in (AC). Also, as

$$(1 - \sigma_0)\varepsilon^{\alpha} < C_1 \varepsilon^{\alpha},$$

we have

$$F(D^2u_{\varepsilon}) = \beta_{\varepsilon}(u_{\varepsilon}) = \gamma u_{\varepsilon}^{\gamma-1}, \quad \text{in } B_{\rho}(X_0).$$

In conclusion, we obtain that u_{ε} is a f_{ij} -subharmonic function in $B_{\rho}(X_0)$ for $\varepsilon \ll 1$, i.e.,

$$f_{ij}D_{ij}u_{\varepsilon} \geq F(D^2u_{\varepsilon}) - C_F = \gamma u_{\varepsilon}^{\gamma-1} - C_F \geq 0,$$

We are now ready to establish the first Hausdorff type estimate for the level surface $\{u_{\varepsilon} \sim \varepsilon^{\alpha}\}$.

Lemma 7.3. *Given a subdomain $\Omega' \Subset \Omega$, there exists a constant C depending on Ω' and universal parameters such that, for $X_0 \in \Omega' \cap \{u_{\varepsilon} > \varepsilon^{\alpha}\}$, with $u_{\varepsilon}(X_0) = C_1 \varepsilon^{\alpha}$, with $C_1 > 1$ and $\varepsilon \ll \rho$ for ρ universally small, there holds*

$$\int_{\{C_1 \varepsilon^{\alpha} < u_{\varepsilon} < \mu^{\alpha}\} \cap B_{\rho}(X_0)} u_{\varepsilon}^{-\gamma} |\nabla u_{\varepsilon}|^2 dX \leq C \mu \rho^{N-1},$$

for a.e. $0 < \rho \ll 1$.

Proof. The proof starts off by verifying that for ε and ρ universally small, the following differential inequality holds:

$$(7.1) \quad \sum_{ij} f_{ij} D_{ij}(u_\varepsilon^{\frac{1}{\alpha}}) \geq 0 \quad \text{in } \{u_0 > 0\} \cap B_\rho(X_0).$$

where $\mathcal{F} = (f_{ij})_{ij}$, as in Definition (7.1). To show such an estimate, we argue as follows: fix a non-singular linear operator $A: \mathbb{R}^N \rightarrow \mathbb{R}^N$. We compute

$$(7.2) \quad \begin{aligned} \sum_{ij} f_{ij} D_{ij}(u_\varepsilon^{\frac{1}{\alpha}}) &= \frac{1}{\alpha} u_\varepsilon^{\frac{1}{\alpha}-1} \text{Tr}(A^{-1} \mathcal{F} (A^{-1})^T D^2(u_\varepsilon \circ A)) \circ A^{-1} \\ &+ \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1 \right) u_\varepsilon^{\frac{1}{\alpha}-2} \text{Tr}(A^{-1} \mathcal{F} (A^{-1})^T \nabla(u_\varepsilon \circ A) \otimes \nabla(u_\varepsilon \circ A)) \circ A^{-1}. \end{aligned}$$

In addition, $u_\varepsilon \circ A$ solves the following uniform elliptic fully nonlinear equation

$$F_A(D^2 v) = \gamma v^{\gamma-1},$$

where the operator F_A is given by

$$F_A(\mathcal{M}) := F((A^{-1})^T \mathcal{M} A^{-1}).$$

Easily one verifies that F_A is in fact uniformly elliptic, with the the same ellipticity constants as F . Thus, by optimal regularity, Theorem 4.1, we obtain

$$|\nabla(u_\varepsilon \circ A)|^2 \leq C(u_\varepsilon \circ A)^\gamma.$$

and so,

$$(7.3) \quad \frac{1}{\alpha} \left(\frac{1}{\alpha} - 1 \right) |\nabla(u_\varepsilon \circ A)|^2 \geq C(u_\varepsilon \circ A)^\gamma.$$

From the structural assumption (AC),

$$(7.4) \quad \begin{aligned} \text{Tr}((A^{-1})^T \mathcal{F} A^{-1} D^2(u_\varepsilon \circ A)) &\geq F((A^{-1})^T D^2(u_\varepsilon \circ A) A^{-1}) - C_F \\ &\geq F((D^2 u_\varepsilon) \circ A) - C_F \\ &\geq \gamma(u_\varepsilon \circ A)^{\gamma-1} - C_F. \end{aligned}$$

Thus, if we select A as to satisfies

$$\mathcal{F} = \frac{1}{C} A A^T$$

where $C > 0$ is the constant of inequality (7.3), and combine 7.3, 7.4 and 7.2, we end up with

$$\sum_{ij} f_{ij} D_{ij}(u_\varepsilon^{\frac{1}{\alpha}}) \geq \frac{1}{\alpha} u_\varepsilon^{\frac{1}{\alpha}-1} \left(\frac{\gamma}{2} u_\varepsilon^{\gamma-1} - C_F \right).$$

Finally from Remark 7.2, we deduce that for ε and ρ universally small, the differential inequality (7.1) indeed holds true.

We now continue with the proof of Lemma 7.3. Define the following cut off function,

$$\Phi = \begin{cases} \varepsilon \sqrt[\alpha]{C_1} & \text{in } \{u_\varepsilon \leq C_1 \varepsilon^\alpha\}; \\ u_\varepsilon^{\frac{1}{\alpha}} & \text{in } \{C_1 \varepsilon^\alpha < u_\varepsilon \leq \mu^\alpha\}; \\ \mu & \text{in } \{u_\varepsilon > \mu^\alpha\}. \end{cases}$$

Clearly we have

$$(7.5) \quad \int_{\{C_1 \varepsilon^\alpha < u_\varepsilon \leq \mu^\alpha\} \cap B_\rho(X_0)} f_{ij}(u_\varepsilon^{\frac{1}{\alpha}})_i \cdot (u_\varepsilon^{\frac{1}{\alpha}})_j dX = \int_{B_\rho(X_0)} f_{ij} \Phi_i(u_\varepsilon^{\frac{1}{\alpha}})_j dX$$

Standard integrations by parts yield

$$(7.6) \quad \begin{aligned} \int_{B_\rho(X_0)} f_{ij} \Phi_i(u_\varepsilon^{\frac{1}{\alpha}})_j dX &= - \int_{B_\rho(X_0)} \Phi \cdot f_{ij}(u_\varepsilon^{\frac{1}{\alpha}})_i dX \\ &+ \frac{1}{\rho} \int_{\partial B_\rho(X_0)} f_{ij} \Phi(u_\varepsilon^{\frac{1}{\alpha}})_i \cdot (x^j - x_0^j) d\mathcal{H}^{N-1}. \end{aligned}$$

Therefore, from the differential inequality established in (7.1), we conclude

$$\int_{\{C_1 \varepsilon^\alpha < u_\varepsilon \leq \mu^\alpha\} \cap B_\rho(X_0)} f_{ij}(u_\varepsilon^{\frac{1}{\alpha}})_i (u_\varepsilon^{\frac{1}{\alpha}})_j dX \leq \frac{1}{\rho} \int_{\partial B_\rho(X_0)} \Phi \cdot f_{ij}(u_\varepsilon^{\frac{1}{\alpha}})_i \cdot (x^j - x_0^j) d\mathcal{H}^{N-1}.$$

Passing the derivatives through, we can further write the above estimate as

$$\int_{\{C_1 \varepsilon^\alpha < u_\varepsilon \leq \mu^\alpha\} \cap B_\rho(X_0)} f_{ij} u_\varepsilon^{-\gamma} D_i u_\varepsilon D_j u_\varepsilon dx \leq \frac{\alpha}{\rho} \int_{\partial B_\rho(X_0)} \Phi \cdot f_{ij} u_\varepsilon^{-\frac{\gamma}{2}} D_i u_\varepsilon \cdot (x^j - x_0^j) d\mathcal{H}^{N-1}.$$

Finally, from uniform ellipticity and optimal regularity of u_ε , we derive

$$\int_{\{C_1 \varepsilon^\alpha < u_\varepsilon \leq \mu^\alpha\} \cap B_\rho(X_0)} u^{-\gamma} |\nabla u|^2 dX \leq C \mu \rho^{N-1},$$

as desired. \square

For the next result, let us recall the following classical notation: given a set $G \subset \mathbb{R}^N$, we will denote

$$\mathcal{N}_\delta(G) := \{X \in \mathbb{R}^N \mid \text{dist}(X, G) < \delta\}.$$

In the sequel we show the main step towards uniform bounds of the \mathcal{H}^{N-1} -Hausdorff measure of the level-surfaces $\{u_\varepsilon > \varepsilon^\alpha\}$.

Lemma 7.4. *Fixed $\Omega' \Subset \Omega$, there exists a constant C^* that depends only on Ω' and universal parameters such that if,*

$$C^*\mu \leq 2\rho \leq \frac{\text{dist}(\Omega', \partial\Omega)}{10}$$

then, for $\mu, \varepsilon > 0$ universally small and $\mu \ll \rho$, for ρ also universally small, we have

$$\mathcal{L}^N(\{C_1\varepsilon^\alpha < u_\varepsilon < \mu^\alpha\} \cap B_\rho(X_0)) \leq \bar{C}\mu\rho^{N-1},$$

where again $\bar{C} = \bar{C}(\Omega')$ depends only on Ω' and universal constants and $X_0 \in \Omega' \cap \partial\{C_1\varepsilon^\alpha < u_\varepsilon < \mu^\alpha\}$, with $d_\varepsilon(X_0) \leq \frac{\text{dist}(\Omega', \partial\Omega)}{10}$ and $C_1 > 1$.

Proof. Let $\{B_j\}$ be a finite family of balls covering $\partial\{C_1\varepsilon^\alpha < u_\varepsilon\} \cap B_\rho(X_0)$, with radius constant equal to $C^*\mu$ and center $X_j \in \partial\{C_1\varepsilon^\alpha < u_\varepsilon\} \cap B_\rho(X_0)$, where C^* will be chosen *a posteriori*. By Heine-Borel Lemma, there exists a universal constant m such that

$$\sum_j \chi_{B_j} \leq m.$$

We can assure that

$$\bigcup_j B_j \subset \left[\mathcal{N}_{\frac{d}{8}}(\Omega') \cap B_{4\rho}(X_0) \right].$$

where $d := \text{dist}(\Omega', \partial\Omega)$. As in the proof of Lemma 7.3, we consider

$$\Phi = \begin{cases} \varepsilon^{\frac{1}{\alpha}C_1} & \text{in } \{u_\varepsilon \leq C_1\varepsilon^\alpha\}; \\ u_\varepsilon^{\frac{1}{\alpha}} & \text{in } \{C_1\varepsilon^\alpha < u_\varepsilon \leq \mu^\alpha\}; \\ \mu & \text{in } \{u_\varepsilon > \mu^\alpha\}. \end{cases}$$

We now claim that is possible to find, for each j , balls B_j^1 and B_j^2 both contained in B_j , satisfying:

(1) the radius of B_j^1 and B_j^2 are in order μ (up to universal contraction)

(2) $\Phi \geq \sqrt[\alpha]{\frac{3}{4}}\mu$ in B_j^1 and $\Phi \leq \sqrt[\alpha]{\frac{2}{3}}\mu$ in B_j^2 .

To show the above claim, we argue as follows: take $X_1 \in \frac{1}{4}\overline{B_j}$, such that

$$u_\varepsilon(X_1) = \sup_{\frac{1}{4}B_j} u_\varepsilon.$$

By strong nondegeneracy,

$$u_\varepsilon(X_1) \geq c_0 \left(\frac{C^*\mu}{4} \right)^\alpha \geq \mu^\alpha,$$

if $C^* \gg 1$ is chosen universally large enough. From Corollary (4.2), given $X \in B_j$, we obtain

$$(7.7) \quad u(X) \geq \mu^\alpha - C_2 \left[\sup_{B_j} u_\varepsilon \right]^{\frac{\gamma}{2}} \cdot |X_1 - X| - C_2 |X_1 - X|^\alpha.$$

By the fact that $u_\varepsilon(X_j) = C_1 \varepsilon^\alpha < \mu^\alpha$ and employing once more Corollary 4.2, we reach

$$\begin{aligned} \sup_{B_j} u_\varepsilon &\leq u_\varepsilon(X_j) + C_2 C^\star u_\varepsilon(X_j)^{\frac{\gamma}{2}} \mu + C_2 (C^\star \mu)^\alpha \\ &\leq (1 + C_2 C^\star + C_2 (C^\star)^\alpha) \mu^\alpha. \end{aligned}$$

Combining the above estimate with (7.7) and taking

$$|X - X_1| < \frac{1}{2C_2} (1 + C_2 C^\star + C_2 (C^\star)^\alpha)^{-\frac{\gamma}{2}} \mu,$$

we obtain

$$\Phi^\alpha(X) = u_\varepsilon(X) \geq \frac{3}{4} \mu \quad \text{in } B_j^1 := B_{r_j^1}(X_1)$$

where $r_j^1 := \tilde{C}_1 \mu$ and

$$\tilde{C}_1 := \frac{1}{4C_2} (1 + C_2 C^\star + C_2 (C^\star)^\alpha)^{-\frac{\gamma}{2}},$$

is a universal constant. To finish up the proof of this first statement, we just choose C^\star large enough as to

$$\tilde{C}_1 \ll C^\star \implies B_j^1 \subset B_j.$$

Notice again that such a selection is universal. Similarly, for $B_j^2 := B_{r_j^2}(X_j)$ where $r_j^2 := \tilde{C}_2 \mu \ll C^\star \mu$, we have

$$\Phi(X)^\alpha = u_\varepsilon \leq \frac{2}{3} \mu^\alpha.$$

From property (2) proven above, assures the existence of a universal constant $\kappa > 0$ such that, for each $j \in \mathbb{N}$

$$|\Phi - m_j| > \kappa \mu,$$

in at least one the two balls $B_j^1, B_j^2 \subset B_j$, where

$$m_j := \int_{B_j} \Phi(X) dX.$$

Thus, by classical Poincaré inequality in balls, we derive

$$\kappa^2 \mu^2 \leq \frac{1}{|B_j|} \int_{B_j} |\Phi - m_j|^2 dX \leq C_3 \mu^2 \frac{1}{|B_j|} \int_{B_j} |\nabla \Phi|^2 dX,$$

which in turn gives

$$\int_{\{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\} \cap B_j} u_\varepsilon^{-\frac{\gamma}{2}} |\nabla u_\varepsilon|^2 dx \geq C_4 |B_j|.$$

In addition, by nondegeneracy, for all $Y \in \{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\} \cap B_\rho(X_0)$, we have

$$C_5 d_\varepsilon(Y)^\alpha \leq u_\varepsilon(Y) \leq \mu^\alpha.$$

Hence

$$\{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\} \cap B_\rho(X_0) \subset \mathcal{N}_{\frac{1}{C_6} \mu}(\partial\{C_1 \varepsilon^\alpha < u_\varepsilon\} \cap B_{2\rho}(X_0)),$$

for $C_6 = \sqrt[\alpha]{C_5}$. Thus, for $\mu \ll \rho$, and $C^* \gg 1$, both universal, we reach

$$\{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\} \cap B_\rho(X_0) \subset \bigcup 2B_j \subset B_{4\rho}(X_0).$$

Finally, applying Lemma (7.3) and taking into account the inclusion above, we estimate

$$\begin{aligned} C_7 \mu \rho^{N-1} &\geq \int_{B_{4\rho}(X_0) \cap \{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\}} u_\varepsilon^{-\frac{\gamma}{2}} |\nabla u_\varepsilon|^2 dx \\ &\geq \frac{1}{m} \sum \int_{2B_j \cap \{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\}} u_\varepsilon^{-\frac{\gamma}{2}} |\nabla u_\varepsilon|^2 dx \\ &\geq \frac{C_4}{m} \sum |B_j| \\ &\geq \frac{C_4}{m} |B_\rho(X_0) \cup \{C_1 \varepsilon^\alpha < u_\varepsilon < \mu^\alpha\}|, \end{aligned}$$

where C_7 and C_4 are universal constants, which completes the proof of Lemma. \square

In the sequel, we recall the definition of δ -density.

Definition 7.5. Given an open subset G of \mathbb{R}^N , we say that G has the δ -density property in Ω for $0 < \delta < 1$, if there exists $\tau > 0$ such that

$$\frac{\mathcal{L}^N(B_\delta(X) \cap A)}{\mathcal{L}^N(B_\delta(X))} \geq \tau$$

for all $X \in \partial G \cap \Omega$. If the property above is valid for any $0 < \delta < 1$, we say that G has *uniform density in Ω along ∂G* .

Here we state a, by now, classical result from measure theory.

Lemma 7.6. *Given an open set $A \Subset \Omega$, there holds:*

a) *If there exists δ such that A has the δ -density property, then there exists a constant $C = C(\tau, N)$, where:*

$$|\mathcal{N}_\delta(\partial A) \cap B_\rho(X)| \leq \frac{1}{2^N \tau} |\mathcal{N}_\delta(\partial A) \cap B_\rho(X) \cap A| + C \delta \rho^{N-1}$$

with $X \in \partial A \cap \Omega$ and $\delta \ll \rho$.

b) *If A has uniform density in Ω along A , then $|\partial A \cap \Omega| = 0$.*

We are ready to state and prove the main result of this section.

Theorem 7.7. *Given $\Omega' \Subset \Omega$ there exists a universal constant $C = C(\Omega') > 0$, such that*

$$\mathcal{L}^N(\mathcal{N}_\mu(\{C_1\varepsilon^\alpha < u_\varepsilon\}) \cap B_\rho(X_0)) \leq C\mu\rho^{N-1},$$

whenever, $C_1 > 1$, $X_0 \in \Omega' \cap \partial\{C_1\varepsilon^\alpha < u_\varepsilon\}$, $d_\varepsilon(X_0) < \frac{1}{10}\text{dist}(\Omega', \partial\Omega)$, $\mu \ll \rho$ with ρ universally small and $C_1\varepsilon^\alpha < \mu^\alpha$. In particular,

$$\mathcal{H}^{N-1}(\partial\{C_1\varepsilon^\alpha < u_\varepsilon\} \cap B_\rho(X_0)) \leq C\rho^{N-1}.$$

Proof. Take $\mu = \delta$, is as in the statement of Corollary (5.6). We have,

$$\frac{\mathcal{L}^N(B_\delta(X) \cap \{u_\varepsilon > C_1\varepsilon^\alpha\})}{\mathcal{L}^N(B_\delta(X))} \geq C_2,$$

for $X \in \{u_\varepsilon > C_1\varepsilon^\alpha\}$. We conclude that, $\partial\{u_\varepsilon > C_1\varepsilon^\alpha\}$ has the δ -density property, and by Lemma 7.6, for a universal constant $M > 0$, there holds

$$(7.8) \quad \begin{aligned} \mathcal{L}^N(\mathcal{N}_\delta(\partial\{u_\varepsilon > C_1\varepsilon^\alpha\}) \cap B_\rho(X_0)) &\leq \frac{1}{2^N C_2} \mathcal{L}^N(\mathcal{N}_\delta(\partial\{u_\varepsilon > C_1\varepsilon^\alpha\}) \cap B_\rho(X_0) \cap \{u_\varepsilon > C_1\varepsilon^\alpha\}) \\ &\quad + M\delta\rho^{N-1}. \end{aligned}$$

From Corollary 4.2, given $Y \in \mathcal{N}_\delta(\partial\{u_\varepsilon > C_1\varepsilon^\alpha\}) \cap B_\rho(X_0) \cap \{u_\varepsilon > C_1\varepsilon^\alpha\}$ and $Z \in \partial\{u_\varepsilon > C_1\varepsilon^\alpha\}$, we can estimate

$$\begin{aligned} u(Y) &\leq u(Z) + C_3 u(Z)^{\frac{\gamma}{2}} |Z - Y| + C_4 |Z - Y|^\alpha \\ &\leq \mu^\alpha + C_3 \mu^{\alpha\frac{\gamma}{2}} \delta + C_3 \delta^\alpha \\ &\leq D\mu^\alpha, \end{aligned}$$

where the last inequality, follows from $C_1\varepsilon^\alpha < \mu$ and $\delta = C\mu$. We have verified there exists $D > 0$ universal, such that

$$(7.9) \quad \mathcal{N}_\delta(\partial\{u_\varepsilon > C_1\varepsilon^\alpha\}) \cap B_\rho(X_0) \cap \{u_\varepsilon > C_1\varepsilon^\alpha\} \subset \{C_1\varepsilon^\alpha < u_\varepsilon < D\mu^\alpha\} \cap B_\rho(X_0).$$

Finally, from Lemma (7.4), we conclude

$$(7.10) \quad \mathcal{L}^N(\{C_1\varepsilon^\alpha < u_\varepsilon < D\mu^\alpha\} \cap B_\rho(X_0)) \leq C_4\mu\rho^{N-1},$$

thus, combining (7.8), (7.9) and (7.10) we reach,

$$\mathcal{L}^N(\mathcal{N}_\mu(\{u_\varepsilon > C_1\varepsilon^\alpha\}) \cap B_\rho(X_0)) \leq C_4\mu\rho^{N-1}.$$

To conclude the proof of the \mathcal{H}^{N-1} Hausdorff measure estimate, let $\{B_j\}$ be a covering of $\partial\{C_1\varepsilon^\alpha < u_\varepsilon\} \cap B_\rho(X_0)$, where each ball be centered in $\partial\{C_1\varepsilon^\alpha < u_\varepsilon\} \cap B_\rho(X_0)$ with radius μ . We can write

$$\bigcup B_j \subset \mathcal{N}_\mu(\{C_1\varepsilon^\alpha < u_\varepsilon\}) \cap B_{\rho+\mu}(X_0).$$

Thus there exist dimensional constants $C_5, C_6 > 0$, such that

$$\begin{aligned}
 \mathcal{H}_\mu^{N-1}(\partial\{C_1\varepsilon^\alpha < u_\varepsilon\} \cap B_\rho(X_0)) &\leq C_5 \sum \text{Area}(\partial B_j) \\
 &= \frac{C_5}{\mu} \sum \mathcal{L}^N(B_j) \\
 &\leq \frac{C_6}{\mu} \mathcal{L}^N(\mathcal{N}_\mu(\{C_1\varepsilon^\alpha < u_\varepsilon\}) \cap B_{\rho+\mu}(X_0)) \\
 &\leq C_6 C_4 (\rho + \mu)^{N-1} = C_6 C_4 \rho^{N-1} + o(1)
 \end{aligned}$$

Letting $\mu \rightarrow 0$, we finish the proof of the Theorem. \square

8 Limiting free boundary problem

In this Section, we address the fully nonlinear free boundary problem obtained by letting $\varepsilon \rightarrow 0$. The ultimate goal is to find a solution to the free boundary problem (1.2) that enjoys all the desired analytic and geometric properties.

Our analysis starts off by the compactness of minimal solutions to Equation (E_ε) . In fact, Theorem (4.1) implies, for any $\Omega' \Subset \Omega$,

$$(8.1) \quad \|u_\varepsilon\|_{C^{1, \frac{\gamma}{2-\gamma}}(\Omega')} \leq K,$$

for some K independent of ε . Therefore, $\{u_\varepsilon\}_{\varepsilon>0}$ is a compact sequence in $C_{\text{loc}}^1(\Omega)$ and up to a subsequence,

$$(8.2) \quad \lim_{\varepsilon \rightarrow 0} u_\varepsilon =: u_0.$$

Clearly, from (8.1), the limiting function u_0 lies in $C_{\text{loc}}^{1, \frac{\gamma}{2-\gamma}}(\Omega)$. This Section is devoted to the study of the limiting function u_0 and the free boundary problem it solves.

For the readers convenience, let us hereafter set up the following notations that we will use throughout this Section:

$$\begin{aligned}
 \{u_0 > 0\} &:= \{x \in \Omega \mid u_0(x) > 0\}, \\
 \mathfrak{F}(u_0) &:= \partial\{u_0 > 0\} \cap \Omega, \\
 d_0(X) &:= \text{dist}(X, \mathfrak{F}(u_0)).
 \end{aligned}$$

Next Theorem recovers the fully nonlinear equation satisfies by u_0 within its positive set as well as its precise growth behavior near the free boundary, $\mathfrak{F}(u_0)$.

Theorem 8.1. *The limiting function u_0 defined in (8.2) is a viscosity solution to*

$$(8.3) \quad F(D^2u) = \gamma u^{\gamma-1} \quad \text{in } \{u > 0\}.$$

Moreover, for a fixed $\Omega' \Subset \Omega$, there exists a constant $C = C(\Omega')$ that depends on Ω' and universal constants such that for any $X \in \Omega' \cap \{u_0 > 0\}$, there holds

$$Cd_0(X)^\alpha \leq u_0(X) \leq C^{-1}d_0(X)^\alpha,$$

whenever $d_0(X) \leq \frac{\text{dist}(\Omega', \partial\Omega)}{4}$. In particular, u_0 is a solution to the free boundary problem (1.2).

Proof. Let us fix a point $X_0 \in \{u_0 > 0\}$ and let $u_0(X_0) := \sigma > 0$. By continuity $u_0 \geq \frac{1}{2}\sigma$ in $B_\rho(X_0)$ for some $\rho > 0$. Since $u_\varepsilon \rightarrow u_0$ uniformly over compact sets, for $\varepsilon \ll 1$ we have

$$u_\varepsilon \geq \frac{1}{8}\sigma > (1 + \sigma_0)\varepsilon^\alpha.$$

That is, u_ε satisfies

$$F(D^2u_\varepsilon) = \gamma u_\varepsilon^{\gamma-1} \quad \text{in } B_{\frac{1}{2}\rho}(X_0).$$

By the stability of viscosity solutions under uniform limits, we conclude u_0 is indeed a viscosity solution to Equation (8.3).

Let us now turn our attention to the growth rate controls. For that, fix $X_0 \in \Omega' \cap \{u_0 > 0\}$, with $d_0(X_0) \leq \frac{1}{4}\text{dist}(\Omega', \partial\Omega)$ and label $u_0(X_0) = s > 0$. For $\varepsilon \ll 1$ we have

$$u_\varepsilon(X_0) \geq \frac{s}{2} > \varepsilon^\alpha.$$

Thus, according to Corollary 5.4, we obtain

$$u_\varepsilon(X_0) \geq Cd_\varepsilon(X_0)^\alpha.$$

Let $Y_\varepsilon \in \partial\{u_\varepsilon > \varepsilon^\alpha\}$ be such that $d_\varepsilon(X_0) = |X_0 - Y_\varepsilon|$. By uniform convergence, it clearly follows that $Y_\varepsilon \rightarrow Y_0$ and $u_0(Y_0) = 0$. In conclusion,

$$u_0(X_0) \geq C|X_0 - Y_0|^\alpha \geq Cd_0(X_0)^\alpha.$$

The upper estimate is obtained similarly. □

Strong nondegeneracy property established for the approximating solutions u_ε also passes to the limiting configuration.

Theorem 8.2. *Given $\Omega' \Subset \Omega$, there exist universal constants $C, \rho_0 > 0$, depending only on Ω' and universal constants, such that for any $X \in \Omega' \cap \overline{\{u_0 > 0\}}$, $\rho \leq \rho_0$ and $d_0(X) < \frac{\text{dist}(X, \partial\Omega')}{2}$, there holds*

$$C^{-1}\rho^\alpha \leq \sup_{B_\rho(X)} u_0 \leq u_0(X) + Cu_0(X)^{\gamma/2}\rho + C\rho^\alpha$$

The proof of Theorem 8.2 is very similar to the one presented for Theorem 8.1 and therefore, we shall omit the details. Next we show the approximating configurations $\{u_\varepsilon\}$ converge to the limiting one, $\{u_0 > 0\}$ in the Hausdorff metric.

Theorem 8.3. *Given $\delta > 0$ and $\varepsilon \ll 1$, the following inclusions hold:*

$$\{u_0 > 0\} \cap \Omega' \subset \mathcal{N}_\delta(\{u_\varepsilon > C_1 \varepsilon^\alpha\}) \cap \Omega' \quad \text{and} \quad \{u_\varepsilon > C_1 \varepsilon^\alpha\} \cap \Omega' \subset \mathcal{N}_\delta(\{u_0 > 0\}) \cap \Omega'.$$

Proof. We will show only the last inclusion, as the first follows similarly. Suppose, for the purpose of contradiction, that such inclusion is false. There would exist, therefore, $\delta_0 > 0$ and a sequence of points $\{X_\varepsilon\}$, satisfying

- a) $X_\varepsilon \in \Omega' \cap \{u_\varepsilon > C_1 \varepsilon^\alpha\}$;
- b) $\text{dist}(X_\varepsilon, \{u_0 > 0\}) > \delta_0$;
- c) $X_\varepsilon \rightarrow X_0$, and $\text{dist}(X_0, \{u_0 > 0\}) > \delta_0$.

From property c) $u_0(X_0) = 0$. However, by strong non-degeneracy, Theorem (8.2), for each ε , we can find $Z_\varepsilon \in B_{\frac{1}{2}\delta_0}(X_\varepsilon)$, such that

$$(8.4) \quad u_\varepsilon(Z_\varepsilon) = \sup_{B_{\frac{1}{2}\delta_0}(X_\varepsilon)} u_\varepsilon \geq C \delta_0^\alpha$$

As $\varepsilon \rightarrow 0$, up to a subsequence, $Z_\varepsilon \rightarrow Z_0$. However, from (8.4) and $u_0(Z_0) > 0$ and by property c) above $Z_0 \in \{u_0 = 0\}$, which is a contradiction. \square

It also follows as in Corollary 5.6 that the set $\{u_0 > 0\}$ has uniform positive density along the free boundary $\mathfrak{F}(u_0)$.

Theorem 8.4. *Given $\Omega' \Subset \Omega$ there exists a constant $0 < c \leq 1$, depending on Ω' and universal parameters, such that*

$$\frac{\mathcal{L}^N(B_\delta(X) \cap \{u_0 > 0\})}{\mathcal{L}^N(B_\delta(X))} \geq c,$$

for all $X \in \mathfrak{F}(u_0) \cap \Omega'$.

The proof of Theorem 8.4 is similar to the one presented for Corollary 5.6 and therefore we omit the details.

As a consequence of the analysis carried out in Section 6, we will show that a *clean* Harnack inequality is valid near the free boundary $\mathfrak{F}(u_0)$. As mentioned in that Section, such a result is quite surprising at first view, as the nonlinear source of the equation is of order $\sim u^{\gamma-1}$ and thus it blows up near the boundary of the quenching region.

Theorem 8.5 (Harnack Inequality for tangential balls). *Let $X_0 \in \{u_0 > 0\}$ and $d := d_0(X_0)$. Then, there exist a universal constant $C > 0$ such that*

$$\sup_{B_{\frac{d}{2}}(X_0)} u_0 \leq C \inf_{B_{\frac{d}{2}}(X_0)} u_0.$$

Proof. Let $\xi_0, \xi_1 \in \overline{B_{\frac{d}{2}}(X_0)}$, be such that

$$\inf_{B_{\frac{d}{2}}(X_0)} u_0 = u_0(\xi_0) \quad \text{and} \quad \sup_{B_{\frac{d}{2}}(X_0)} u_0 = u_0(\xi_1).$$

Since $d_0(\xi_0) \geq \frac{d}{2}$, by Theorem 8.2, there holds

$$(8.5) \quad u_0(\xi_0) \geq C_1 d^\alpha.$$

On the other hand, from Theorem 8.2,

$$u_0(\xi_1) \leq u_0(X_0) + C_1 u_0(X_0)^{\frac{\gamma}{2}} d + C_1 d^\alpha.$$

As in the proof of Corollary 4.2, that can be easily adapted to this present case, for any $Y \in \partial \{u_0 > 0\}$, we have

$$u_0(X_0) \leq u_0(Y) + C_1 u_0(Y)^{\frac{\gamma}{2}} d + C_1 d^\alpha \leq C_1 d^\alpha.$$

Thus,

$$\sup_{B_{\frac{d}{2}}(X_0)} u_0 \leq C_2 \inf_{B_{\frac{d}{2}}(X_0)} u_0.$$

for a constant $C_2 > 0$ that does not depend of u_0 . \square

As in the proof of Corollary 6.1, we can establish a lower bound for solid integrals for u_0 : for all $X_0 \in \partial \{u_0 > 0\} \cap \Omega'$

$$(8.6) \quad C_1 \rho^\alpha \leq \int_{B_\rho(X_0)} u_0 dx,$$

where $C_1 = C_1(\Omega') > 0$. Next we establish upper and lower control on spherical integrals of u_0 .

Theorem 8.6. *Given $\Omega' \Subset \Omega$, there exists a universal constant $C = C(\Omega')$, such that for all $X_0 \in \partial \{u_0 > 0\} \cap \Omega'$,*

$$C^{-1} \rho^\alpha \leq \int_{\partial B_\rho(X_0)} u_0 d\mathcal{H}^{N-1} \leq C \rho^\alpha.$$

Proof. The upper estimate follows directly from Corollary (8.2). We will show the lower bound by means of contradiction. Suppose the lower inequality is not valid. There would then exist $\rho_m > 0$ and $X_m \in \partial \{u_0 > 0\}$, such that

$$(8.7) \quad \frac{1}{\rho_m^\alpha} \int_{\partial B_{\rho_m}(X_m)} u_0 d\mathcal{H}^{N-1} = o(1),$$

as $m \rightarrow \infty$. Clearly, (8.7) implies

$$(8.8) \quad \frac{1}{\rho_m^\alpha} \int_{\partial B_{r_{\rho_m}}(X_m)} u_0 d\mathcal{H}^{N-1} = o(1),$$

for all $0 < r \leq 1$. Define

$$v_m(X) := \rho_m^{-\alpha} u_0(X_m + \rho_m X).$$

Up to a subsequence, v_m converges uniformly over compact subsets of \mathbb{R}^N , to a function v_0 . Furthermore,

$$(8.9) \quad F_m(D^2 v_m) = \gamma v_m^{\gamma-1} \quad \text{in } \{v_m > 0\},$$

where $F_m(\mathcal{M}) := \rho_m^\alpha F(\rho_m^{-\alpha} \mathcal{M})$. For any $0 < r \leq 1$,

$$\int_{\partial B_{r\rho_m}(X_m)} u_0(Y) d\mathcal{H}^{N-1} = \int_{\partial B_r(0)} u_0(X_m + \rho_m X) d\mathcal{H}^{N-1} = \rho_m^\alpha \int_{\partial B_r(0)} v_m(X) d\mathcal{H}^{N-1}.$$

Thus, by (8.8), letting $m \rightarrow \infty$, yields

$$\int_{\partial B_r(0)} v_0 d\mathcal{H}^{N-1} = \rho_m^{-\alpha} \int_{\partial B_{r\rho_m}(X_m)} u_0 d\mathcal{H}^{N-1} = 0, \quad \forall 0 < r \leq 1.$$

Therefore, $v_0 \equiv 0$ in B_1 which contradicts (8.6) properly scaled to v_m . \square

9 Geometric estimates of the free boundary

In this final Section we obtain further fine geometric-measure properties of the free boundary $\mathfrak{F}(u_0)$. As in Section 7, here we shall work under the addition structural assumption (AC). The first result we show concerns the local finiteness of the \mathcal{H}^{N-1} -Hausdorff measure of the free boundary $\mathfrak{F}(u_0)$.

Theorem 9.1. *Given $\Omega' \Subset \Omega$ there exists a constant $C = C(\Omega') > 0$, depending on Ω' and universal constants, such that*

$$\mathcal{L}^N(\mathcal{N}_\mu(\{u_0 > 0\}) \cap B_\rho(X_0)) \leq C\mu\rho^{N-1},$$

whenever, $X_0 \in \Omega' \cap \partial\{u_0 > 0\}$, $d_0(X_0) < \frac{1}{10}\text{dist}(\Omega', \partial\Omega)$, $\mu \ll \rho$ and ρ is universally small. In particular,

$$\mathcal{H}^{N-1}(B_\rho(X_0) \cap \mathfrak{F}(u_0)) \leq C\rho^{N-1}.$$

Proof. From Theorem 7.7 and Theorem 8.3, we have for $\varepsilon \ll 1$

$$|\mathcal{N}_{2\mu}(\{u_\varepsilon > C_1 \varepsilon^\alpha\}) \cap B_\rho(X_0)| \leq C\mu\rho^{N-1}$$

and

$$\{u_0 > 0\} \cap B_\rho(X_0) \subset \mathcal{N}_\mu(\{u_\varepsilon > C_1 \varepsilon^\alpha\}) \cap B_\rho(X_0).$$

Easily we show

$$\mathcal{N}_\mu(\{u_0 > 0\}) \cap B_\rho(X_0) \subset \mathcal{N}_{2\mu}(\{u_\varepsilon > C_1 \varepsilon^\alpha\}) \cap B_\rho(X_0),$$

which give us the estimate desired. \square

A consequence of Theorem 9.1 is that the limiting region $\{u_0 > 0\}$ has locally finite perimeter. The key final result we will show here states that the reduced free boundary, $\partial_{\text{red}}\{u_0 > 0\}$ has total measure. More importantly, we prove that around points Z of the reduced free boundary, there holds

$$\mathcal{H}^{N-1}(B_\rho(Z) \cap \mathfrak{F}(u_0)) \sim \rho^{N-1}.$$

In particular the free boundary has a theoretical measure outward unit vector for \mathcal{H}^{N-1} almost all points in $\mathfrak{F}(u_0)$.

Theorem 9.2. *Given $\Omega' \Subset \Omega$, there exists a positive constant $C = C(\Omega')$, that depends only on Ω' and universal constants, such that for any ball $B_\rho(X_0)$, with ρ universally small, centered at a free boundary point $x_0 \in \partial\{u_0 > 0\}$, there holds*

$$C^{-1}\rho^{N-1} \leq \mathcal{H}^{N-1}(\partial_{\text{red}}\{u_0 > 0\} \cap B_\rho(X_0)) \leq C\rho^{N-1}.$$

In particular,

$$\mathcal{H}^{N-1}(\partial\{u_0 > 0\} \setminus \partial_{\text{red}}\{u_0 > 0\}) = 0.$$

Proof. The estimate from above follows from Theorem 9.1. It remains to verify the estimate by below. Fixed X_0 , let us define the normalized function $v_0 : B_1 \rightarrow \mathbb{R}$ by

$$v_0(X) := \frac{u_0(X_0 - \rho X)}{\rho^\alpha}.$$

Arguing as in the proof of Theorem (7.3), for ρ universally small, we conclude,

$$(9.1) \quad L(v_0^{\frac{1}{\alpha}}) := \rho^{2-\alpha} \sum_{ij} f_{ij} D_{ij}(u_0^{\frac{1}{\alpha}}) \geq 0 \quad \text{in} \quad \{v_0 > 0\} \cap B_1.$$

Our next step is to furnish an appropriate special barrier. Let ψ be a nonnegative smooth function in B_1 , with $\psi \equiv 1$ in $B_{1/5}$ and $\psi \equiv 0$ outside $B_{1/4}$. Let Φ be the solution to the following boundary value problem

$$\begin{cases} L\Phi = -\psi & \text{in } B_1 \\ \Phi = 0 & \text{on } \partial B_1. \end{cases}$$

From classical elliptic regularity theory, Φ is smooth and, in particular, for any $0 < \alpha < 1$,

$$(9.2) \quad \|\Phi\|_{C^\alpha(B_{1/2})} \leq C_1,$$

by a universal constant $C_1 > 0$. Also by maximum principle $\Phi > 0$ in B_1 and by Hopf maximum principle,

$$(9.3) \quad f_{ij} \partial_i \Phi v_j \geq C_2 > 0, \quad \text{along } \partial B_1,$$

where v_j is the j -th coordinate of the outward normal vector to $\partial B_1(0)$. Applying generalized Gauss-Green formula, we derive

$$(9.4) \quad \begin{aligned} \int_{\{v_0 > 0\} \cap B_1} \left\{ \Phi L(v_0^{\frac{1}{\alpha}}) - v_0^{\frac{1}{\alpha}} L\Phi \right\} dx &= \int_{\partial_{\text{red}}\{v_0 > 0\} \cap B_1} \left\{ \Phi f_{ij} \partial_i (v_0^{\frac{1}{\alpha}}) - v_0^{\frac{1}{\alpha}} f_{ij} \partial_i \Phi \right\} \eta_j d\mathcal{H}^{N-1} \\ &\quad - \int_{\{v_0 > 0\} \cap \partial B_1} v_0^{\frac{1}{\alpha}} f_{ij} \partial_i \Phi v_j d\mathcal{H}^{N-1}. \end{aligned}$$

Since $\Phi L(v_0^{\frac{1}{\alpha}}) \geq 0$, there holds

$$(9.5) \quad \int_{\{v_0 > 0\} \cap B_1} \left\{ \Phi L(v_0^{\frac{1}{\alpha}}) - v_0^{\frac{1}{\alpha}} L \Phi \right\} dx \geq \int_{B_1} \psi v_0^{\frac{1}{\alpha}} dx \geq \int_{B_{1/5}} v_0^{\frac{1}{\alpha}} dx.$$

Also from uniform gradient bounds of v_0 , ellipticity and (9.2) we estimate

$$(9.6) \quad \left| \int_{\partial_{\text{red}}\{v_0 > 0\} \cap B_1} \Phi f_{ij} \partial_i (v_0^{\frac{1}{\alpha}}) \eta_j d\mathcal{H}^{N-1} \right| \leq C_1 \mathcal{H}^{N-1}(\partial_{\text{red}}\{v_0 > 0\} \cap B_1).$$

In addition, clearly,

$$(9.7) \quad \int_{\partial_{\text{red}}\{v_0 > 0\} \cap B_1} v_0^{\frac{1}{\alpha}} f_{ij} \partial_i \Phi \eta_j d\mathcal{H}^{N-1} = 0,$$

and by (9.3),

$$(9.8) \quad \int_{\{v_0 > 0\} \cap \partial B_1} v_0^{\frac{1}{\alpha}} f_{ij} \partial_i \Phi v_j d\mathcal{H}^{N-1} \geq 0.$$

Combining (9.4), (9.5), (9.6) and (9.7), we deduce

$$(9.9) \quad \int_{B_{1/5}} v_0^{\frac{1}{\alpha}} dx \leq C_1 \mathcal{H}^{N-1}(\partial_{\text{red}}\{v_0 > 0\} \cap B_1).$$

On the other hand, by non-degeneracy, as in proof of Theorem 6.1, there holds

$$(9.10) \quad \int_{B_{1/5}(0)} v_0^{\frac{1}{\alpha}} dx \geq C_3.$$

for a positive universal constant c . Finally from (9.9) and (9.10) we conclude

$$\mathcal{H}^{N-1}(\partial_{\text{red}}\{v_0 > 0\} \cap B_1) \geq c_0,$$

for a universal constant c_0 and the estimate by below is proven. The total measure of the reduced free boundary follows now by classical considerations. \square

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